A computational study of a class of recursive inequalities

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The proof mining program aims to give a computational interpretation to prima facie non-effective proofs through the application of tools from logic. In recent years, proof mining has enjoyed many successes in nonlinear analysis, with logical tools being used to extract very uniform bounds (e.g. bounds independent of the space). In this work, we present a contribution to the proof mining of nonlinear analysis.

Recursive inequalities play a big role in nonlinear analysis. A common way they are used is in establishing the convergence of an iteratively defined sequence of elements in some space to a point satisfying some properties. A simple example of this can be seen through the Banach fixed point theorem where, it can be shown that, for a contraction mapping T with constant $c \in [0, 1)$ and x^* a fixed point of T, the distance $\mu_n := d(T^n x_0, x^*)$ satisfies $\mu_{n+1} \leq c\mu_n$ and thus converges to 0.

In our work, we study the convergence properties sequences of nonnegative real numbers $\{\mu_n\}$ and $\{\beta_n\}$ satisfying,

$$\mu_{n+1} \leq \mu_n - \alpha_n \beta_n + \gamma_n \tag{1}$$

with $\{\alpha_n\}$ a nonnegative sequence of real numbers with a divergent sum and $\{\gamma_n\}$ a nonnegative sequence of real numbers that converges to 0. This recursive inequality features in numerous optimization problems in nonlinear analysis. Typically α_n represents some step size for an algorithm and γ_n represents an error term.

One can easily produce examples where the condition that $\gamma_n \to 0$ is not enough to deduce the convergence of either $\{\mu_n\}$ or $\{\beta_n\}$. Thus, in the literature this condition is usually strengthened to one of the two cases:

(I) $\sum_{i=0}^{\infty} \gamma_i < \infty$

(II) $\gamma_n/\alpha_n \to 0$ as $n \to \infty$.

We study each of these cases in turn and obtain quantitative results about the convergence of $\{\mu_n\}$ and $\{\beta_n\}$ by producing computable rates of convergences, in some cases.

It is a known result of Specker [1] that it is not always possible to obtain a computable rate of convergence for converging sequences of computable numbers. In our work we also produce similar negative results. In scenarios where it is impossible to produce a computable rate of convergence we obtain, instead, a rate of metastablity. This is a functional $\Phi : \mathbb{Q}_+ \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ satisfying,

$$\forall \varepsilon \in \mathbb{Q}_+ \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \leq \Phi(\varepsilon, g) \, \forall k \in [n, n + g(n)](|a_k - a| \leq \varepsilon) \tag{2}$$

where, $[a, a+b] := \{a, a+1, \ldots, a+b\}$.

The idea of metastability comes from logic. If one takes takes the Herbrand normal form of the definition of convergence, we obtain a finitary version of this principle (in the sense of Tao [8]). A rate of metastability will be a computable interpretation to this definition and can be recognised as being a solution to the so-called 'no-counterexample interpretation' of the definition of convergence [2,3]. Obtaining rates of metastability using proof theoretic techniques is a standard result in applied proof theory (e.g.[4,5,6]).

After an abstract study of recursive inequalities, we discuss how our results about the convergence properties of real numbers have application in nonlinear analysis. We adapt the work of Alber et al. in [7], to produce a general gradient descent algorithm and rates of metastability for the convergence of our algorithm to a solution. Furthermore, we are able to pin point the exact ineffective principles which stopped the authors of [7] from being able to produce explicit rates of convergences for their algorithm. In addition, we demonstrate how our work generalises known results in the proof mining literature such as the study of Mann schemes for asymptotically weakly contractive mappings [9] and in the study of set values accretive operators ([10] for example).

Alongside this theoretical work, we have also started a Lean library [1] devoted to implementing quantitative results that use recursive inequalities. This work will be useful as it would allow us to have implemented a large class of core lemmas used in both in the formalization of nonlinear analysis and proof theoretic applications. Our formalization project is still very much in its early stages, with only a handful of known rates of convergences and metastabilities from the literature, to date, being verified. In addition, we have also implemented a key construction, from computable analysis, of a sequence of rational numbers converging to zero without a computable rate of convergence. This sequence has been adapted allover the applied

[1] https://github.com/mneri123/Proof-mining-

proof theory literature to produce negative results, of the type previously discussed. I shall discuss interesting aspects of the formalization that has been done so far and also outline future directions for research in both implementation and potentially automated reasoning.

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