An Introduction to Proof Mining Metatheorems Through the Lens of Probability Theory

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6^{th} Southern and Midlands Logic Seminar, University of Oxford

Applied proof theory

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- Explicit notable case studies came from Kreisel and Luckhardt in the 60s and late 80s, respectively (Littlewood's theorem and Roth's theorem).
- Due to work from Kohlenbach and his collaborators in the late 90s and early 2000s, a more systematic approach was developed with plenty of success in extracting the quantitative content from proofs of results in approximation theory and analysis.
- This approach was then logically substantiated by the first so-called proof mining metatheorems by Kohlenbach, appearing in 2005. The importance of these metatheorems was twofold.

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Applied proof theory in one slide

- Firstly, they demonstrated that the previous successes in proof mining were not a happy coincidence but actually a part of a general phenomenon that could be explained using logic.
- Secondly, the systems used to describe such phenomena are powerful enough to formalise a large amount of mathematics, which demonstrated that this approach could be taken much further than it already had.

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Structure of talk

Since the first metatheorems, many more have appeared. These results expand the work of Kohlenbach into new areas of mathematics. These metatheorems either expand the logical tools used in the original or give insights into the area of study that allow for one to formalise a useful fragment of the area in an extension of the original system.

In this talk, we give an introduction to the logical systems and tools of proof mining by looking at the most recent of such metatheorems, which treat probability theory. This was a joint project with Nicholas Pischke.

The structure of the talk will be as follows:

- Motivation for the system.
- The system and logical metatheorem.
- Achievements of the system (if there is time).

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Motivation for the system.

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The theory of contents

The first problem one faces in trying to axiomatise a theory of probability that is amenable to the methods of proof mining is how to deal with the countable union axioms (even the defining property of a countable union).

These axioms are naturally not admissible in the context of the usual approach to proof mining metatheorems. Thus, we are stuck with axiomatising those properties of infinite unions that are and hoping this is enough to do proof mining.

We can do a lot without infinite unions.

It turns out that there is a rich theory known as the theory of contents (or charges), where one essentially does normal probability theory without infinite unions and countable additivity. Furthermore, we observed that (at a glance) the few results from Avigad et al. in proof mining in probability theory appeared to only deal with finite unions, thus fuelling our confidence that formalising the theory of charges would lead to great success.

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The system and logical metatheorem

The basic system

As common in proof mining, our formal systems will be extensions of $\mathcal{A}^\omega = \mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC} + \mathsf{DC}$, which denotes (a weakly extensional variant of) Peano arithmetic in all finite types together with a few choice principles, with such a system providing a formalization of classical analysis in all finite types.

We now sketch $\mathcal{A}^\omega$.

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We now sketch $\mathcal{A}^\omega$.

A sketch of $\mathcal{A}^\omega$: Typing

We have a set of all finite types T

$$0 \in T, \quad \rho, \tau \in T \to \rho(\tau) \in T.$$

Objects of type 0 are interpreted as natural numbers natural numbers and objects of type $\rho(\tau)$ represent mappings from objects of type τ to objects of type ρ. Furthermore, we denote pure types by natural numbers by setting $n + 1 := 0(n)$.

For example, an object of type $2 = 0(1)$ should be interpreted as a functional mapping function between the natural numbers to the natural numbers.

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A sketch of $\mathcal{A}^\omega$: Rules, relations and constants

The theory $\mathcal{A}^\omega$ is an extension of many-sorted classical logic with variables taking types from T and constants: 0 (zero), S (successor), Π, Σ (combinators), $\underline{R}$ (simultaneous primitive recursion in all types). The only primitive relation is equality at type 0 (denoted by $=_0$).

Higher-type equality is only defined as an abbreviation via recursion with

$$x^{\tau(\xi)} =_{\tau(\xi)} y^{\tau(\xi)} := \forall z^\xi \left(xz =_\tau yz \right).$$

Crucially, we do not have the full extensionality principle

$$\forall x^{\tau(\rho)}, y^\rho, y'^\rho \left(y =_\rho y' \to xy =_\tau xy' \right)$$

because this would not allow for a result on program extraction. Instead, it only contains the quantifier-free extensionality rule

$$\frac{A_0 \to s =_\rho t}{A_0 \to r[s/x^\rho] =_\tau r[t/x^\rho]}$$

where A_0 is a quantifier-free formula, s and t are terms of type ρ and r is a term of type τ.

A sketch of $\mathcal{A}^\omega$: Rules, relations and constants

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A sketch of $\mathcal{A}^\omega$: Non-logical axioms

The first group of axioms allow us to do very basic arithmetic:

$=_0$ is an equivalence relation.

The usual successor axiom.

An axiom schema of complete induction for all formulas of the language.

We then have an axiom allowing us to do simultaneous primitive recursion.

$$\begin{cases} (R_i)_{\underline{\rho}} 0 \underline{y}\underline{z} =_{\rho_i} y_i \\ (R_i)_{\underline{\rho}} (Sx)\underline{y}\underline{z} =_{\rho_i} z_i(\underline{R}_{\underline{\rho}} x\underline{y}\underline{z})x. \end{cases}$$

Lastly, we have combinator axioms that allow us to define lambda abstraction.

Namely, for any term t of type τ and any variable x of type ρ, we can construct a term $\lambda x.t$ of type $\tau(\rho)$ such that the free variables of $\lambda x.t$ are exactly those of t without x and so that

$$\text{WE-PA}^\omega \vdash (\lambda x.t)(s) =_\tau t[s/x].$$

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A quick sketch of $\mathcal{A}^\omega$: Axioms of choice

We have the following choice principles:

$$\forall \underline{x} \exists \underline{y} A_0(\underline{x}, \underline{y}) \to \exists \underline{Y} \forall \underline{x} A_0(\underline{x}, \underline{Y}\underline{x}) \qquad \text{(QF-AC)}$$

with A_0 quantifier-free and where the types of the variable tuples $\underline{x}, \underline{y}$ are arbitrary.

$$\forall x^0, \underline{y}^{\underline{\rho}} \exists \underline{z}^{\underline{\rho}} A(x, \underline{y}, \underline{z}) \to \exists \underline{f}^{\underline{\rho}(0)} \forall x^0 A(x, \underline{f}(x), \underline{f}(S(x))) \qquad (\mathrm{DC}^{\underline{\rho}})$$

A may now be arbitrary.

$\mathrm{DC}^{\underline{\rho}}$ implies countable choice, and so we have arbitrary comprehension over natural numbers. Therefore, full second-order arithmetic (in the sense of that used in reverse maths) can be embedded in $\mathcal{A}^\omega$.

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A quick sketch of $\mathcal{A}^\omega$: Representation of real numbers

Pairs of naturals represent the rationals through a canonical paring function definable in the system. Furthermore, the operations $+_\mathbb{Q}$, $\cdot_\mathbb{Q}$, $|\cdot|_\mathbb{Q}$, etc., are then primitive recursively definable and $=_\mathbb{Q}$, $<_\mathbb{Q}$, etc., are definable via quantifier-free formulas.

Real numbers are represented via Cauchy sequences of rational numbers with a fixed Cauchy modulus 2^{-n} (objects of type 1). The operations, $+_\mathbb{R}$, $\cdot_\mathbb{R}$, $|\cdot|_\mathbb{R}$, etc., are primitive recursively definable through closed terms and the relations $=_\mathbb{R}$ and $<_\mathbb{R}$, etc., are representable via formulas of the underlying language. Unlike the rationals, these relations are not decidable but are given by Π^0_1- and Σ^0_1-formulas, respectively.

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A system for sample space and events

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$\mathcal{F}^\omega$ denotes the system for reasoning about the sample space and events. More precisely, $\mathcal{F}^\omega$ is an extension of $\mathcal{A}^\omega$ augmented with the new set of types (where all the respective constants and axioms now are allowed to also refer to these new types, if applicable) $T^{\Omega,S}$ defined by

$$0, \Omega, S \in T^{\Omega,S}, \quad \rho, \tau \in T^{\Omega,S} \to \rho(\tau) \in T^{\Omega,S}.$$

Along with the new constants

- eq of type $0(\Omega)(\Omega)$;
- $\in$ of type $0(S)(\Omega)$;
- $\cup$ of type $S(S)(S)$;
- $(\cdot)^c$ of type $S(S)$;
- $\emptyset$ of type S;
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Provable properties of sample space and events

We define equality on S via the following abbreviation: for A^S and B^s, we define

$$A =_S B :\equiv \forall x^\Omega (x \in A \leftrightarrow x \in B).$$

$=_S$ is, probably, an equivalence relation.

We can also introduce the abbreviation

$$A \subseteq_S B :\equiv \forall x^\Omega (x \in A \to x \in B)$$

for A, B of type S and show that $\subseteq_S$ forms a partial order with respect to equality defined by $=_S$.

Furthermore, the operations $\cup$ and $(\cdot)^c$ are provably extensional with respect to $=_S$.

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Provable properties of sample space and events

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System for probability contents

$\mathcal{F}[\mathbb{P}]^\omega$ denotes the formal system for probability contents on algebras. It is an extension of $\mathcal{F}^\omega$ with a new constant symbol $\mathbb{P}$ of type $1(S)$ along with four axioms.

The first two are purely universal:

$$\forall A^S (0 \leq_\mathbb{R} \mathbb{P}(A) \leq_\mathbb{R} 1),$$
$$\mathbb{P}(\emptyset) =_\mathbb{R} 0.$$

The final axiom we need to fully characterise contents on algebras is finite additivity.

$$\forall A^S, B^S (A \cap B =_S \emptyset \to \mathbb{P}(A \cup B) =_\mathbb{R} \mathbb{P}(A) + \mathbb{P}(B))$$

however, this is not purely universal and is instead equivalent (over $\mathcal{F}^\omega$ extended with the constant $\mathbb{P}$) to the following generalized Π_3-sentence:

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However, in order to ease the formal development of our system, we do not actually include the above axiom to our, but instead, add the following generalized additivity law, which holds for probability contents on algebras:

$$\forall A^S, B^S(\mathbb{P}(A \cup B) =_{\mathbb{R}} \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)).$$

This statement is purely universal and thus immediately admissible.

The other property we add is that of the monotonicity of i.e.

$$\forall A^S, B^S \left(A \subseteq_S B \to \mathbb{P}(A) \leq_{\mathbb{R}} \mathbb{P}(B)\right).$$

Similar to above, this statement is equivalent to the following (generalized) Π_3-statement

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Following this axiomatization fully captures the theory of finitely additive measure spaces. For example, we can prove:

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The metatheorem

Gödel's Dialectica combined with the negative translation

The Dialectica interpretation takes a formula A in the language of WE-PA$^\omega$ and produces a formula $A^D = \exists \underline{x} \forall \underline{y} A_D(\underline{x}, \underline{y})$, with the only free variables of A_D being that of A, $\underline{x}$ and $\underline{y}$.

We do not spell out the full interpretation, but we note that over a semi-intuitionistic fragment of WE-PA$^\omega$, A and A^D are equivalent.

Furthermore, for formulas A that are provable in this same semi-intuitionistic fragment, one can obtain realisers for the A^D.

For example, if $A := \forall x \exists y A(x, y)$ then $A^D = \exists F \forall x A(x, F(x))$.

In order to get access to full classical logic, we introduce the negative translation of A (of Kuroda), which we note by A'. This takes a formula provable in WE-PA$^\omega$ and outputs a formula provable in the semi-intuitionistic fragment that allows for the extraction of realisers via the Dialectica interpretation.

Thus, combining these two interpretations gives one the ability to extract realisers of formulas provable in WE-PA$^\omega$.

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The soundness theorem

In fact, more is true. We can also get realisers for proofs that use choice principles. We have the following soundness theorem:

Let $\mathcal{P}$ be a set of universal sentences and let $A(\underline{a})$ be an arbitrary formula (with only the variables $\underline{a}$ free) in the language of $\mathsf{WE\text{-}PA}^\omega$. Then the rule

$$\begin{cases} \mathsf{WE\text{-}PA}^\omega + \mathsf{QF\text{-}AC} + \mathsf{DC} + \mathcal{P} \vdash A(\underline{a}) \Rightarrow \\ \mathsf{WE\text{-}PA}^\omega + (\mathrm{BR}) + \mathcal{P} \vdash \forall \underline{a}, \underline{y} (A')_D(\underline{ta}, \underline{y}, \underline{a}) \end{cases}$$

holds where $\underline{t}$ is a tuple of closed terms of $\mathsf{WE\text{-}PA}^\omega + (\mathrm{BR})$ which can be extracted from the respective proof and (BR) is the schema of *(simultaneous) bar-recursion* of Spector.

What is important about (the proof of) this result to us is that it is also true for any extension of the language of $\mathsf{WE\text{-}PA}^\omega$ that introduces new types and constants, together with any number of additional universal axioms in that language. Thus, this result holds for $\mathcal{F}^\omega$.

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A quick introduction to bar recursion

Given a well founded tree T, $M : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, $N : \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, and $\sigma \in \mathbb{N}^{<\mathbb{N}}$ one can define the function $\tilde{B}$

$$\tilde{B}(M,N)(\sigma) := \begin{cases} M(\sigma) & \text{if } \sigma \notin T \\ \\ N(\sigma, \lambda n.\tilde{B}(M,N)(\sigma * < n >)) & \text{otherwise.} \end{cases}$$

A quick introduction to bar recursion

Now, for $K : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, (a form of) bar recursion is defined through a function B satisfying:

$$B(M, N, K)(\sigma) := \begin{cases} M(\sigma) & \text{if } \exists i \leq |\sigma| \, K(\sigma|_i) \\ N(\sigma, \lambda n. B(M, N, K)(\sigma * < n >)) & \text{otherwise.} \end{cases}$$

WE-PA$^{\omega}$ + (BR) is an extension of WE-PA$^{\omega}$ with a new constant allowing for (a weaker form of) bar recursion as defined above (generalised to all finite types and allowed to be done simultaneously).

However, this principle is set theoretically false! A natural condition on K, for bar recursion to be well defined is that K is continuous. That is,

$$\forall f \in \mathbb{N}^{\mathbb{N}} \, \exists n \in \mathbb{N} \, \forall g \in \mathbb{N}^{\mathbb{N}} \, (f|_n = g|_n \to K(f) = K(g)).$$

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$$\forall f \in \mathbb{N}^{\mathbb{N}}\, \exists n \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, (f|_n = g|_n \to K(f) = K(g)).$$

How about $\mathcal{F}^\omega[\mathbb{P}]$?

Recall that this system had the axiom,

$$\forall A^S, B^S \left(A \subseteq_S B \to \mathbb{P}(A) \leq_{\mathbb{R}} \mathbb{P}(B)\right)$$

that was not universal.

We define the relation $\leq_\rho$ by recursion on the type via

1. $x \leq_0 y := x \leq_0 y$.
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Models

For such an extension of $\mathcal{F}^\omega$, by using the monotone functional interpretation due to Kohlenbach, we can also obtain a proof-theoretical program extraction result with a (slightly more complex) verifying theory that still includes bar recursion.

We are still faced with the problem of the truth of these extractions. Bar recursion does not hold in the natural model of WE-PA$^\omega$. Furthermore, it's extension to $\mathcal{F}^\omega[\mathbb{P}]$ also fails to hold in the natural models, which we call $\mathcal{S}^{\omega,\Omega,S}$, defined via $\mathcal{S}_0 := \mathbb{N}$, $\mathcal{S}_\Omega := \Omega$, $\mathcal{S}_S := S$ and

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It is a remarkable result of Bezman that the following structure:

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is a model of $\mathsf{WE\text{-}PA}^\omega + (\mathrm{BR})$ that contains discontinuous functionals! The relation $\gtrsim$ is called strong majorization, and the model is known as the structure of all hereditarily strongly majorizable set-theoretic functionals of finite types.

We would like to extend this model to the context of probability spaces. In order to do this, we must project our abstract types down to types in T. That is, define $\widehat{\tau} \in T$, given $\tau \in T^{\Omega,S}$, by recursion on the structure via

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Program extraction for $\mathcal{F}^\omega[\mathbb{P}]$

By adapting Bezem's proof, one can show that $\mathcal{M}^{\omega,\Omega,S}$ models $\mathcal{F}^{\omega-} + (\text{BR})$.

Where $\mathcal{F}^{\omega-}$ is $\mathcal{F}^\omega$ without QF-AC and DC.

To deal with $\mathcal{F}^\omega[\mathbb{P}]$, extend $\mathcal{F}^{\omega-}$ to a system $\mathcal{C}^\omega$ as follows:

Add a new constant X to the language of $\mathcal{F}^{\omega-}$ (as well as $\mathbb{P}$), along with the universal axiom

$$X \leq_{\Omega(S(S))} (\lambda A, B.c_\Omega) \wedge \forall A^S, B^S\, (\mathbb{P}(A) >_{\mathbb{R}} \mathbb{P}(B) \to X(A,B) \in A \wedge X(A,B) \notin B).$$

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Furthermore, we have the soundness theorem also holding for $\mathcal{C}^\omega$, that is,

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However, now the term t depends on this new constant X introduced to the language.

We can show the following:
For any closed term t of $\mathcal{C}^\omega + (\mathrm{BR})$, one can construct a closed term t^* of $\mathcal{A}^\omega + (\mathrm{BR})$ such that

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So, if we are just interested in bounds for realisers, this can be done in the language of WE-PA$^\omega$.

Thus, not only will the bounds be independent of X, but they will be independent of all the new constants we added to the language of WE-PA$^\omega$!!

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Small and accessible types

Before we can state our metatheorem, we must introduce some terminology.

We define the degree of a type in T by induction as follows:

$deg(0) := 0$ and $deg(\tau(\rho)) = \max\{deg(\tau), deg(\rho) + 1\}$.

We call a type ξ *of degree n* if $\xi \in T$ and it has degree $\leq n$.

We call $\xi \in T^{\Omega,S}$ *small* if it is of the form $\xi = \xi_0(0)\dots(0)$ for $\xi_0 \in \{0, \Omega, S\}$ (including $0, \Omega, S$) and call it *admissible* if it is of the form $\xi = \xi_0(\tau_k)\dots(\tau_1)$ where each τ_i is small and $\xi_0 \in \{0, \Omega, S\}$ (also including $0, \Omega, S$).

The importance of these definitions, in the proof of the metaheorem, is that it turns out that if ρ is small $\mathcal{M}_\rho = S_\rho$ and if ρ is accessible $\mathcal{M}_\rho \subseteq S_\rho$

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We call $\xi \in T^{\Omega,S}$ *small* if it is of the form $\xi = \xi_0(0)\ldots(0)$ for $\xi_0 \in \{0, \Omega, S\}$ (including $0, \Omega, S$) and call it *admissible* if it is of the form $\xi = \xi_0(\tau_k)\ldots(\tau_1)$ where each τ_i is small and $\xi_0 \in \{0, \Omega, S\}$ (also including $0, \Omega, S$).

The importance of these definitions, in the proof of the metaheorem, is that it turns out that if ρ is small $\mathcal{M}_\rho = S_\rho$ and if ρ is accessible $\mathcal{M}_\rho \subseteq S_\rho$

We can now state our program extraction theorem.

The metatheorem

Let τ be admissible, δ be of degree 1 and s be a closed term of $\mathcal{C}^\omega$ of type $\sigma(\delta)$ for admissible σ. Let $B_\forall(x,z,u)/C_\exists(x,z,v)$ be $\forall$-/$\exists$-formulas of $\mathcal{F}^\omega[\mathbb{P}]$ with only $x,y,z,u/x,y,z,v$ free.

If

$$\mathcal{F}^\omega[\mathbb{P}] \vdash \forall x^\delta \forall y \leq_\sigma s(x) \forall z^\tau \left(\forall u^0 B_\forall(x,y,z,u) \to \exists v^0 C_\exists(x,y,z,v) \right),$$

then one can extract a partial functional $\Phi : \mathcal{S}_\delta \times \mathcal{S}_{\widehat{\tau}} \rightharpoonup \mathbb{N}$ which is total and (bar-recursively) computable on $\mathcal{M}_\delta \times \mathcal{M}_{\widehat{\tau}}$ and such that for all $x \in \mathcal{S}_\delta, z \in \mathcal{S}_\tau, z^* \in \mathcal{S}_{\widehat{\tau}}$, if $z^* \gtrsim z$, then

$$\mathcal{S}^{\omega,\Omega,S} \models \forall y \leq_\sigma s(x)(\forall u \leq_0 \Phi(x,z^*) B_\forall(x,y,z,u) \to \exists v \leq_0 \Phi(x,z^*) C_\exists(x,y,z,v))$$

holds.

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Thank you!